MATHEMATICS

Relations and Functions

Top Concepts in Relations

1. **Introduction to Relation and no. of relations**
   - A relation R between two non-empty sets A and B is a subset of their Cartesian product $A \times B$.
   - If A = B, then the relation R on A is a subset of $A \times A$.
   - The total number of relations from a set consisting of m elements to a set consisting of n elements is 2^{mn}.
   - If (a, b) belongs to R, then a is related to b and is written as 'a R b'. If (a, b) does not belong to R, then a is not related to b and it is written as 'a $\not R$ b'.

2. **Domain, Co-domain and Range of a Relation**
   Let R be a relation from A to B. Then the 'domain of R' $\subset$ A and the 'range of R' $\subset$ B. Co-domain is either set B or any of its superset or subset containing range of R.

3. **Types of Relations**
   - A relation R in a set A is called an **empty** relation if no element of A is related to any element of A, i.e. $R = \phi \subset A \times A$.
   - A relation R in a set A is called a **universal** relation if each element of A is related to every element of A, i.e. $R = A \times A$.

4. A relation R on a set A is called:
   a.**Reflexive**, if $(a, a) \in R$ for every $a \in A$.
   b.**Symmetric**, if $(a_1, a_2) \in R$ implies that $(a_2, a_1) \in R$ for all $a_1, a_2 \in A$.
   c.**Transitive**, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$ for all $a_1, a_2, a_3 \in A$.

5. **Equivalence Relation**
   - A relation R in a set A is said to be an **equivalence relation** if R is reflexive, symmetric and transitive.
   - An empty relation R on a non-empty set X (i.e. 'a R b' is never true) is not an equivalence relation, because although it is vacuously symmetric and transitive, but it is not reflexive (except when X is also empty).

6. Given an arbitrary equivalence relation R in a set X, R divides X into mutually disjoint subsets S_i called partitions or subdivisions of X provided:

a. All elements of S_i are related to each other for all i.

b. No element of S_i is related to any element of S_j if $i \neq j$.

c. $\bigcup_{i=1}^{n} S_j$ = X and $S_i \cap S_j = \phi$ if $i \neq j$.

The subsets S_j are called equivalence classes.

7. **Union, Intersection and Inverse of Equivalence Relations**

   a. If R and S are two equivalence relations on a set A, $R \cap S$ is also an equivalence relation on A.

   b. The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.

   c. The inverse of an equivalence relation is an equivalence relation.

Top Concepts in Functions

1. **Introduction to functions**

   A function from a non-empty set A to another non-empty set B is a correspondence or a rule which associates every element of A to a unique element of B written as $f : A \to B$ such that f(x) = y for all $x \in A$, $y \in B$.

   All functions are relations, but the converse is not true.

2. **Domain, Co-domain and Range of a Function**

- If $f : A \to B$ is a function, then set A is the domain, set B is the co-domain and set $\{f(x) : x \in A\}$ is the range of f.
- The range is a subset of the co-domain.
- A function can also be regarded as a machine which gives a unique output in set B corresponding to each input from set A.
- If A and B are two sets having m and n elements, respectively, then the total number of functions from A to B is n^m.

3. **Real Function**

- A function $f : A \to B$ is called a real-valued function if B is a subset of R.
- If A and B both are subsets of R, then 'f' is called a real function.
- While describing real functions using mathematical formula, x (the input) is the independent variable and y (the output) is the dependent variable.
- The graph of a real function 'f' consists of points whose co-ordinates (x, y) satisfy $y = f(x)$, for all $x \in \text{Domain}(f)$.

4. **Vertical line test**

   A curve in a plane represents the graph of a real function if and only if no vertical line intersects it more than once.

5. One-one Function

- A function $f : A \to B$ is **one-to-one** if for all $x, y \in A$, $f(x) = f(y) \Rightarrow x = y$ or $x \neq y \Rightarrow f(x) \neq f(y)$
- A one-one function is known as an injection or **injective** function. Otherwise, f is called many-one.

6. Onto Function

- A function $f : A \to B$ is an **onto** function, if for each $b \in B$, there is at least one $a \in A$ such that $f(a) = b$, i.e. if every element in B is the image of some element in A, then f is an onto or **surjective** function.
- For an onto function, range = co-domain.
- A function which is both one-one and onto is called a **bijective** function or a bijection.
- A one-one function defined from a finite set to itself is always onto, but if the set is infinite, then it is not the case.

7. Let A and B be two finite sets and $f : A \to B$ be a function.

a. If f is an injection, then $n(A) \leq n(B)$.

b. If f is a surjection, then $n(A) \geq n(B)$.

c. If f is a bijection, then $n(A) = n(B)$.

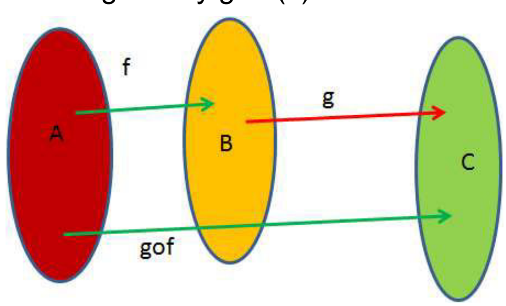
8. If A and B are two non-empty finite sets containing m and n elements, respectively, then

- Number of functions from A to B = n^m
- Number of one-one functions from A to B $= \begin{cases} {}^nC_m \times m!, & \text{if } n \geq m \\ 0, & \text{if } n<m \end{cases}$
- Number of onto functions from A to B = $\begin{cases} \sum_{r=1}^{n} (-1)^{n-r} \; {}^nC_r r^m, & \text{if } m \geq n \\ 0, & \text{if } m<n \end{cases}$
- Number of one-one and onto functions from A to B = $\begin{cases} n!, & \text{if } m=n \\ 0, & \text{if } m \neq n \end{cases}$

9. If a function $f : A \to B$ is not an onto function, then $f : A \to f(A)$ is always an onto function.

10. Composition of Functions

- Let $f : A \to B$ and $g : B \to C$ be two functions. The composition of f and g, denoted by g o f, is defined as the function $g \circ f : A \to C$ and is given by $g \circ f(x) : A \to C$ defined by $g \circ f(x) = g(f(x))\ \forall\ x \in A$.



- Composition of f and g is written as g o f and not f o g.
- g o f is defined if the range of f $\subseteq$ domain of g, and f o g is defined if the range of g $\subseteq$ domain of f.
- Composition of functions is not commutative in general i.e., $f \circ g(x) \neq g \circ f(x)$.
- Composition is associative i.e. if $f : X \to Y$, $g : Y \to Z$ and $h : Z \to S$ are functions, then $h \circ (g \circ f) = (h \circ g) \circ f$.

- The composition of two bijections is a bijection.

11. Inverse of a Function

- Let $f : A \to B$ is a bijection, then $g : B \to A$ is inverse of f if $f(x) = y \Leftrightarrow g(y) = x$ OR $g \circ f = I_A$ and $f \circ g = I_B$
- If $g \circ f = I_A$ and f is an injection, then g is a surjection.
- If $f \circ g = I_B$ and f is a surjection, then g is an injection.

12. Let $f : A \to B$ and $g : B \to C$ be two functions. Then

- $g \circ f : A \to C$ is onto $\Rightarrow g : B \to C$ is onto
- $g \circ f : A \to C$ is one-one $\Rightarrow f : A \to B$ is one-one
- $g \circ f : A \to C$ is onto and $g : B \to C$ is one-one $\Rightarrow f : A \to B$ is onto
- $g \circ f : A \to C$ is one-one and $f : A \to B$ is onto $\Rightarrow g : B \to C$ is one-one

13. Invertible Function

- A function $f : X \to Y$ is defined to be invertible if there exists a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$.
- The function g is called the inverse of f and is denoted by f^{-1}. If f is invertible, then f must be one-one and onto, and conversely, if f is one-one and onto, then f must be invertible.
- If $f : A \to B$ and $g : B \to C$ are one-one and onto, then $g \circ f : A \to C$ is also one-one and onto. But if g o f is one-one, then only f is one-one and g may or may not be one-one. If g o f is onto, then g is onto and f may or may not be onto.
- Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then g o f is also invertible with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
- If f: $R \to R$ is invertible, $f(x) = y$, then $f^{-1}(y) = x$ and $(f^{-1})^{-1}$ is the function f itself.

Binary Operations

1. A binary operation * on a set A is a function from A × A to A.

2. If * is a binary operation on a set S, then S is closed with respect to *.

3. Binary operations on R

- Addition, subtraction and multiplication are binary operations on R, which is the set of real numbers.
- Division is not binary on R; however, division is a binary operation on R − {0}, which is the set of non-zero real numbers.

4. Laws of Binary Operations

- A binary operation $*$ on the set X is called commutative, if $a * b = b * a$, for every $a, b \in X$.
- A binary operation $*$ on the set X is called associative, if $a * (b * c) = (a * b) * c$, for every $a, b, c \in X$.
- An element $e \in A$ is called an **identity** of *A* with respect to * if for each $a \in A$, $a * e = a = e * a$.
- The identity element of (A, *), if it exists, is **unique**.

5. Existence of Inverse

- Given a binary operation $*$ from $A \times A \rightarrow A$ with the identity element e in A, an element $a \in A$ is said to be invertible with respect to the operation $*$, if there exists an element b in A such that $a * b = e = b * a$ and b is called the inverse of a and is denoted by a^{-1}.

6. If the operation table is symmetric about the diagonal line, then the operation is commutative.

*	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

The operation * is commutative.

7. Binary Operation on Natural Numbers

Addition '+' and multiplication '·' on N, the set of natural numbers, are binary operations. However, subtraction '−' and division are not, because $(4, 5) = 4 - 5 = -1 \notin N$ and $4/5 = .8 \notin N$.

8. Number of Binary Operations

- Let S be a finite set consisting of n elements. Then $S \times S$ has n^2 elements.
- The total number of functions from a finite set A to a finite set B is $[n(B)]^{n(A)}$. Therefore, the total number of binary operations on S is n^{n^2}.
- The total number of commutative binary operations on a set consisting of n elements is $n^{\frac{n(n-1)}{2}}$.

MIND MAP : LEARNING MADE SIMPLE

CHAPTER - 1

Relations and Functions

Types of Relations

Empty relations

A relation $R : A \to A$ is empty if $a \not R b \ \forall \ a, b \in A$. $R = \phi \subset A \times A$.
For eg : $R = \{(a,b) : a = b^2\}, A = \{1,5,10\}$

Universal relation

A relation $R : A \to A$ is universal if $a R b \ \forall, a,b \in A, R = A \times A$.
if $R = \phi$, then R is universal.

Trivial Relations

Reflexive relation

A relation $R : A \to A$ is reflexive if $aRa \ \forall \ a \in A$

Symmetric relation

A relation $R : A \to A$ is symmetric if $aRb \Rightarrow bRa \ \forall \ a, b \in A$

Transitive relation

A relation $R : A \times A$ is transitive if $aRb, bRc \Rightarrow aRc \ \forall \ a,b,c \in A$.

+ +

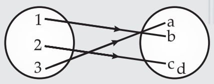
Equivalence relation
(reflexive, symmetric, transitive $e.g.$, Let T = the set of all triangles in a plane and $R : T \to T$ defined by $R = \{(T_1, T_2)\} : T_1$ is congruent to $T_2\}$. Then, R is equivalence.

Types of Functions

One-one (injective)

$f : x \to y$ is one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ $\forall \ x_1, x_2 \in x$. Other wise, f is many- one, f_1 is one-one.

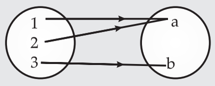
f_1



Onto (surjective)

$f : x \to y$ is onto it for energy $y \in Y, \exists \ x \in X$ S.t. $f(x) = y$, f_2 is onto.

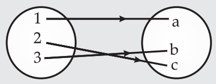
f_2



Bijective

$f : x \to y$ is both one-one and onto, then f is bijective.
f_3 is bijective.

f_3



Composition of functions and Invertible functions

Composition of functions

The composition of functions $f : A \to B$ and $g : B \to C$ is denoted by gof, and is defined as $gof : A \to C$ given by $gof(x) = g(f(x)) \ \forall x \in A$. $e.g.$ let $A = N$ and $f, g : N \to N$ such that $f(x) = x^2$ and $g(x) = x^3$ $\forall \ x \in N$. Then $gof(2) = g(f(2)) = g(2^2)$ $= 4^3 = 64$.

Invertible functions

A function $f : x \to y$ is invertible, if $\exists$ a function $g : y \to x$ such that $gof = I_X$ and $fog = I_Y$. Then, g is the inverse of f. If f is invertible, then it is both one-one and onto and vice-versa. For eg. If $f(x) = x$ and $f : N \to N$, then f is invertible.
Theorem 1 : If $f : x \to y$, $g : y \to z$ and $h : z \to s$ are functions, then $ho(got) = (hog)of$.
Theorem 2 : Let $f : x \to y$ and $g : y \to z$ be two invertible functions, then gof is invertible and $(gof)^{-1} = f^{-1} o \ g^{-1}$.

Binary operations

Definition and its types

A binary operation '*' on a set A is a function $* : A \times A \to$ A denoted by $a * b$ $i.e.$ $\forall$ $a,b, \in$ A, $a * b \in$ A. Commutative if $a * b = b * a \ \forall \ a,b \in A$. Associative if $(a * b) * c = a * (b * c) \ \forall a,b, \in$ A. $e \in A$ is identity if $a * e = a = e * a \ \forall a \in A$. and $b \in A$ is the inverse of $a \in A$, if. $a * b = e = b * a$. Addition is a binary operation on the set of integers.

Important Questions

Multiple Choice questions-

1. Let R be the relation in the set (1, 2, 3, 4}, given by:

R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}.

Then:

(a) R is reflexive and symmetric but not transitive

(b) R is reflexive and transitive but not symmetric

(c) R is symmetric and transitive but not reflexive

(d) R is an equivalence relation.

2. Let R be the relation in the set N given by: R = {(a, b): a = b – 2, b > 6}. Then:

(a) $(2, 4) \in R$

(b) $(3, 8) \in R$

(c) $(6, 8) \in R$

(d) $(8, 7) \in R$.

3. Let A = {1, 2, 3}. Then number of relations containing {1, 2} and {1, 3}, which are reflexive and symmetric but not transitive is:

(a) 1

(b) 2

(c) 3

(d) 4.

4. Let A = (1, 2, 3). Then the number of equivalence relations containing (1, 2) is

(a) 1

(b) 2

(c) 3

(d) 4.

5. Let f: R → R be defined as $f(x) = x^4$. Then

(a) f is one-one onto

(b) f is many-one onto

(c) f is one-one but not onto

(d) f is neither one-one nor onto.

6. Let f: R → R be defined as f(x) = 3x. Then

(a) f is one-one onto

(b) f is many-one onto

(c) f is one-one but not onto

(d) f is neither one-one nor onto.

7. If f: R → R be given by $f(x) = (3 - x^3)^{1/3}$, then $f_o f$ (x) is

(a) $x^{1/3}$

(b) x^3

(c) x

(d) $3 - x^3$.

8. Let f: $R - \{-\frac{4}{3}\}$ → R be a function defined as: $f(x) = \frac{4x}{3x+4}$, $x \neq -\frac{4}{3}$. The inverse of f is map g: Range f → $R - \{-\frac{4}{3}\}$ given by

(a) $g(y) = \frac{3y}{3-4y}$

(b) $g(y) = \frac{4y}{4-3y}$

(c) $g(y) = \frac{4y}{3-4y}$

(d) $g(y) = \frac{3y}{4-3y}$

9. Let R be a relation on the set N of natural numbers defined by nRm if n divides m. Then R is

(a) Reflexive and symmetric

(b) Transitive and symmetric

(c) Equivalence

(d) Reflexive, transitive but not symmetric.

10. Set A has 3 elements, and the set B has 4 elements. Then the number of injective mappings that can be defined from A to B is:

(a) 144

(b) 12

(c) 24

(d) 64

Very Short Questions:

1. If R = {(x, y) : x + 2y = 8} is a relation in N, write the range of R.
2. Show that a one-one function:

   f {1, 2, 3} → {1, 2, 3} must be onto. (N.C.E.R.T.)
3. What is the range of the function $f(x) = \frac{|x-1|}{x-1}$? (C.B.S.E. 2010)
4. Show that the function f : N → N given by f(x) = 2x is one-one but not onto. (N.C.E.R.T.)
5. If f : R → R is defined by f(x) = 3x + 2 find f(f(x)). C.B.S.E. 2011 (F))
6. If $f(x) = \frac{x}{x-1}$, $x \neq 1$ then find fof. (N.C.E.R.T)
7. If f: R → R is defined by $f(x) = (3 - x^3)^{1/3}$, find fof (x)
8. Are f and q both necessarily onto, if gof is onto? (N.C.E.R.T.)

Short Questions:

1. Let A be the set of all students of a Boys' school. Show that the relation R in A given by:

   R = {(a, b): a is sister of b} is an empty relation and the relation R' given by :

   R' = {(a, b) : the difference between heights of a and b is less than 3 metres} is an universal relation. (N.C.E.R.T.)

2. Let f : X → Y be a function. Define a relation R in X given by :

   R = {(a,b):f(a) = f(b)}.

   Examine, if R is an equivalence relation. (N.C.E.R.T.)

3. Let R be the relation in the set Z of integers given by:

   R = {(a, b): 2 divides a – b}.

   Show that the relation R is transitive. Write the equivalence class [0]. (C.B.S.E. Sample Paper 2019-20)

4. Show that the function:

   f : N → N

   given by f(1) = f(2) = 1 and f(x) = x -1, for every x > 2 is onto but not one-one. (N.C.E.R.T.)

5. Find gof and fog, if:

   f : R → R and g : R → R are given by f (x) = cos x and g (x) = $3x^2$. Show that gof ≠ fog. (N. C.E.R. T.)

6. If f(x) = $\frac{4x+3}{6x-4}$, x ≠ $\frac{2}{3}$ find fof(x)

7. Let A = N x N be the set of ail ordered pairs of natural numbers and R be the relation on the set A defined by (a, b) R (c, d) iff ad = bc. Show that R is an equivalence relation.

8. Let f: R → R be the Signum function defined as:

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

   and g : R → R be the Greatest Integer Function given by g (x) = [x], where [x] is greatest integer less than or equal to x. Then does fog and gof coincide in (0,1]?

Long Questions:

1. Show that the relation R on R defined as R = {(a, b):a ≤ b}, is reflexive and transitive but not symmetric.

2. Prove that function f : N → N, defined by f(x) = $x^2 + x + 1$ is one-one but not onto. Find inverse of f : N → S, where S is range of f.

3. Let A = (x ∈Z : 0 ≤ x ≤ 12}.

   Show that R = {(a, b) : a, b ∈ A; |a – b| is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2]. (C.B.S.E 2018)

4. Prove that the function f: [0, ∞) → R given by f(x) = $9x^2 + 6x - 5$ is not invertible. Modify the co-domain of the function f to make it invertible, and hence find f-1. (C.B.S.E. Sample Paper 2018-19

Assertion and Reason Questions-

1. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is also false.

Assertion(A): Let L be the set of all lines in a plane and R be the relation in L defined as R = {(L1, L2): L1 is perpendicular to L2}.R is not equivalence realtion.

Reason (R): R is symmetric but neither reflexive nor transitive

2. Two statements are given-one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes(a), (b), (c) and (d) as given below.

a) Both A and R are true and R is the correct explanation of A.
b) Both A and R are true but R is not the correct explanation of A.
c) A is true but R is false.
d) A is false and R is also false.

Assertion (A): = {(T1, T2): T1 is congruent to T2}. Then R is an equivalence relation.

Reason(R): Any relation R is an equivalence relation, if it is reflexive, symmetric and transitive.

Case Study Questions-

1. Consider the mapping f: A → B is defined by f(x) = x - 1 such that f is a bijection.

Based on the above information, answer the following questions.

(i) Domain of f is:

a) R - {2}
b) R
c) R - {1, 2}
d) R - {0}

(ii) Range of f is:

a) R
b) R - {2}
c) R - {0}
d) R - {1, 2}

(iii) If g: R - {2} → R - {1} is defined by g(x) = 2f(x) - 1, then g(x) in terms of x is:

a. $\frac{x+2}{x}$

b. $\frac{x+1}{x-2}$

c. $\frac{x-2}{x}$

d. $\frac{x}{x-2}$

(iv) The function g defined above, is:

a) One-one
b) Many-one
c) into
d) None of these

(v) A function f(x) is said to be one-one if.

a. $f(x_1) = f(x_2) \Rightarrow -x_1 = x_2$
b. $f(-x_1) = f(-x_2) \Rightarrow -x_1 = x_2$
c. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
d. None of these

2. A relation R on a set A is said to be an equivalence relation on A iff it is:

I. Reflexive i.e., $(a, a) \in R \ \forall \ a \in A$.
II. Symmetric i.e., $(a, b) \in R \Rightarrow (b, a) \in R \ \forall \ a, b \in A$.
III. Transitive i.e., $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R \ \forall \ a, b, c \in A$.

Based on the above information, answer the following questions.

(i) If the relation R = {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)} defined on the set A = {1, 2, 3}, then R is:

a) Reflexive
b) Symmetric
c) Transitive
d) Equivalence

(ii) If the relation R = {(1, 2), (2, 1), (1, 3), (3, 1)} defined on the set A = {1, 2, 3}, then R is:

a) Reflexive
b) Symmetric
c) Transitive
d) Equivalence

(iii) If the relation R on the set N of all natural numbers defined as R = {(x, y): y = x + 5 and x < 4}, then R is:

a) Reflexive
b) Symmetric
c) Transitive
d) Equivalence

(iv) If the relation R on the set A = {1, 2, 3, ..., 13, 14} defined as R = {(x, y): 3x - y = O}, then R is:

a) Reflexive
b) Symmetric
c) Transitive
d) Equivalence

(v) If the relation R on the set A = {I, 2, 3} defined as R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}, then R is:

a) Reflexive only
b) Symmetric only
c) Transitive only
d) Equivalence

Answer Key-

Multiple Choice questions-

(b) R is reflexive and transitive but not symmetric

(c) (6, 8) ∈ R

(a) 1

(b) 2

(d) f is neither one-one nor onto.

(a) f is one-one onto

(c) x

(b) g(y) = $\frac{4y}{4-3y}$

(b) Transitive and symmetric

(c) 24

Very Short Answer:

1. Solution: Range of R = {1, 2, 3}.

   [∵ When x = 2, then y = 3, when x = 4, then y = 2, when x = 6, then y = 1]

2. Solution: Since 'f' is one-one,

   ∴ under 'f', all the three elements of {1, 2, 3} should correspond to three different elements of the co-domain {1, 2, 3}.

   Hence, 'f' is onto.

3. Solution: When x > 1,

   than f(x) = $\frac{x-1}{x-1}$ = 1.

   When x < 1,

   than f(x) = $\frac{-(x-1)}{x-1}$ = -1

   Hence, Rf = {-1, 1}.

4. Solution:

   Let $x_1, x_2 \in N$.

   Now, $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$

⇒ f is one-one.

Now, f is not onto.

∵ For 1 ∈ N, there does not exist any x ∈ N such that f(x) = 2x = 1.

Hence, f is ono-one but not onto.

5. Solution:

   f(f(x)) = 3 f(x) + 2

   = 3(3x + 2) + 2 = 9x + 8.

6. Solution:

$$fof(x) = f(f(x)) = \frac{f(x)}{f(x)-1}$$

$$= \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-x+1}$$

$$= \frac{x}{1} = x.$$

7. Solution:

   $f_o f(x) = f(f(x)) = (3-(f(x))^3)^{1/3}$

   $= (3 - ((3 - x^3)^{1/3})^3)^{1/3}$

   $= (3 - (3 - x^3))^{1/3} = (x^3)^{1/3} = x.$

8. Solution:

   Consider f: {1, 2, 3, 4} → {1, 2, 3, 4}

   and g: {1, 2, 3,4} → {1,2.3} defined by:

   f(1) = 1, f(2) = 2, f(3) = f(4) = 3

   g (1) = 1, g (2) = 2, g (3) = g (4) = 3.

∴ gof = g (f(x)) {1, 2,3}, which is onto

But f is not onto.

[∵ 4 is not the image of any element]

Short Answer:

1. Solution:

(i) Here R = {(a, b): a is sister of b}.

Since the school is a Boys' school,

∴ no student of the school can be the sister of any student of the school.

Thus R = Φ Hence, R is an empty relation.

(ii) Here R' = {(a,b): the difference between heights of a and b is less than 3 metres}.

Since the difference between heights of any two students of the school is to be less than 3 metres,

∴ R' = A x A. Hence, R' is a universal relation.

2. Solution:

For each a ∈ X, (a, a) ∈ R.

Thus R is reflexive. [∵ f (a) = f(a)]

Now (a, b) ∈ R

⇒ f(a) = f(b)

⇒ f(b) = f (a)

⇒ (b, a) ∈ R.

Thus R is symmetric.

And (a, b) ∈ R

and (b, c) ∈ R

⇒ f(a) = f(b)

and f(b) = f(c)

⇒ f(a)= f(c)

⇒ (a, c) ∈ R.

Thus R is transitive.

Hence, R is an equivalence relation.

3. Solution:

Let 2 divide (a – b) and 2 divide (b – c), where a,b,c ∈ Z

⇒ 2 divides [(a – b) + (b – c)]

⇒ 2 divides (a – c).

Hence, R is transitive.

And [0] = {0, ± 2, ± 4, ± 6,…].

4. Solution:

Since f(1) = f(2) = 1,

∴ f(1) = f(2), where 1 ≠ 2.

∴ 'f' is not one-one.

Let y ∈ N, y ≠ 1,

we can choose x as y + 1 such that f(x) = x – 1

= y + 1 – 1 = y.

Also 1 ∈ N, f(1) = 1.

Thus 'f ' is onto.

Hence, 'f ' is onto but not one-one.

5. Solution:

We have:

f(x) = cos x and g(x) = $3x^2$.

∴ gof (x) = g (f(x)) = g (cos x)

$= 3\ (\cos x)^2 = 3 \cos^2 x$

and fog (x) = f(g (x)) = f($3x^2$) = $\cos 3x^2$.

Hence, gof ≠ fog.

6. Solution:

We have: $\frac{4x+3}{6x-4}$ ...(1)

∴ fof(x) - f (f (x))

$$= \frac{4f(x)+3}{6f(x)-4}$$

$$= \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4} \quad [Using\ (1)]$$

$$= \frac{16x+12+18x-12}{24x+18-24x+16}$$

$$= \frac{34x}{34} = x.$$

7. Solution:

Given: (a, b) R (c, d) if and only if ad = bc.

(I) (a, b) R (a, b) iff ab – ba, which is true.

[∵ ab = ba ∀ a, b ∈ N]

Thus, R is reflexive.

(II) (a, b) R (c,d) ⇒ ad = bc

(c, d) R (a, b) ⇒ cb = da.

But cb = be and da = ad in N.

∴ (a, b) R (c, d) ⇒ (c, d) R (a, b).

Thus, R is symmetric.

(III) (a,b) R (c, d)

⇒ ad = bc …(1)

(c, d) R (e,f)

⇒ cf = de … (2)

Multiplying (1) and (2), (ad). (cf) – (be), (de)

⇒ af = be

⇒ (a,b) = R(e,f).

Thus, R is transitive.

Thus, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

8. Solution:

For x ∈ (0,1].

$$(fog)(x) = f(g(x)) = f([x])$$
$$= \begin{cases} f(0)\,; \text{if } 0 < x < 1 \\ f(1)\,; \text{if } x = 1 \end{cases}$$

$$\Rightarrow \quad f(g(x)) = \begin{cases} 0\,; \text{if } 0 < x < 1 \\ 1\,; \text{if } x = 1 \end{cases} \quad ...(1)$$

And (gof) (x) = g(f(x)) = g(1)

[∵ f(x) = 1 ∀ x > 0]

= [1] = 1

⇒ (gof) (x) = 1 ∀ x ∈ (0, 1] …(2)

From (1) and (2), (fog) and (gof) do not coincide in (0, 1].

Long Answer:

1. Solution:

We have: R = {(a, b)} = a ≤ b}.

Since, $a \leq a \ \forall \ a \in R$,

$\therefore (a, a) \in R$,

Thus, R reflexive.

Now, $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow a \leq b$ and $b \leq c$

$\Rightarrow a \leq c$

$\Rightarrow (a, c) \in R$.

Thus, R is transitive.

But R is not symmetric

[$\because (3, 5) \in R$ but $(5, 3) \notin R$ as $3 \leq 5$ but $5 > 3$]

Solution:

Let $x_1, x_2 \in N$.

Now, $f(x_1) = f(x_2)$

$$\Rightarrow x^2{}_1 + x_1 + 1 = x^2{}_2 + x_2 + 1$$
$$\Rightarrow x^2{}_1 + x_1 = x^2{}_2 + x_2$$
$$\Rightarrow (x^2{}_1 - x^2{}_2) + (x_1 - x_2) = 0$$
$$\Rightarrow (x_1 - x_2) + (x_1 + x_2 + 1) = 0$$
$$\Rightarrow x_1 - x_2 = 0 \qquad [\because x_1 + x_2 + 1 \neq 0]$$
$$\Rightarrow x_1 = x_2.$$

Thus, f is one-one.

Let $y \in N$, then for any x,

$f(x) = y$ if $y = x^2 + x + 1$

$\Rightarrow \quad y = \left(x^2 + x + \frac{1}{4}\right) + \frac{3}{4}$

$\Rightarrow \quad y = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

$\Rightarrow \quad x + \frac{1}{2} = \pm\sqrt{y - \frac{3}{4}}$

$\Rightarrow \quad x = \pm\frac{\sqrt{4y-3}}{2} - \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pm\sqrt{4y-3} - 1}{2}$

$\Rightarrow \quad x = \frac{\sqrt{4y-3} - 1}{2}$

$\left[\frac{-\sqrt{4y-3}-1}{2} \notin N \text{ for any value of } y\right]$

Now, for $y = \frac{3}{4}$, $x = -\frac{1}{2} \notin N$.

Thus, f is not onto.

$\Rightarrow$ f(x) is not invertible.

Since, $x > 0$, therefore, $\frac{\sqrt{4y-3}-1}{2} > 0$

$\Rightarrow \sqrt{4y-3} > 1$

$\Rightarrow 4y - 3 > 1$

$\Rightarrow 4y > 4$

$\Rightarrow y > 1$.

Redefining, $f : (0, \infty) \rightarrow (1, \infty)$ makes

$f(x) = x^2 + x + 1$ on onto function.

Thus, f (x) is bijection, hence f is invertible and $f^{-1} : (1, \infty) \rightarrow (1,0)$

$f^{-1}(y) = \frac{\sqrt{4y-3}-1}{2}$

2. Solution:

We have:

R = {(a, b): a, b ∈ A; |a – b| is divisible by 4}.

(1) Reflexive: For any $a \in A$,

$\therefore (a, b) \in R$.

$|a - a| = 0$, which is divisible by 4.

Thus, R is reflexive.

Symmetric:

Let $(a, b) \in R$

$\Rightarrow |a - b|$ is divisible by 4

$\Rightarrow |b - a|$ is divisible by 4

Thus, R is symmetric.

Transitive: Let $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow |a - b|$ is divisible by 4 and $|b - c|$ is divisible by 4

$\Rightarrow |a - b| = 4\lambda$

$\Rightarrow a - b = \pm 4\lambda$ ....(1)

and $|b - c| = 4\mu$

$\Rightarrow b - c = \pm 4\mu$ ....(2)

Adding (1) and (2),

$(a-b) + (b-c) = \pm 4(\lambda + \mu)$

$\Rightarrow a - c = \pm 4 (\lambda + \mu)$

$\Rightarrow (a, c) \in R$.

Thus, R is transitive.

Now, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

(ii) Let 'x' be an element of A such that $(x, 1) \in R$

$\Rightarrow |x - 1|$ is divisible by 4

$\Rightarrow x - 1 = 0, 4, 8, 12, \ldots$

$\Rightarrow x = 1, 5, 9, 13, \ldots$

Hence, the set of all elements of A which are related to 1 is {1, 5, 9}.

(iii) Let $(x, 2) \in R$.

Thus $|x - 2| = 4k$, where $k \le 3$.

$\therefore x = 2, 6, 10.$

Hence, equivalence class [2] = {2, 6, 10}.

3. Solution:

Let $y \in R$.

For any x, f(x) = y if $y = 9x^2 + 6x - 5$

$\Rightarrow y = (9x^2 + 6x + 1) - 6$

$= (3x + 1)^2 - 6$

$$\Rightarrow \quad 3x + 1 = \pm\sqrt{y+6}$$

$$\Rightarrow \quad x = \frac{\pm\sqrt{y+6} - 1}{3}$$

$$\Rightarrow \quad x = \frac{\sqrt{y+6} - 1}{3}$$

$$\left[\because \frac{-\sqrt{y+6} - 1}{3} \notin [0, \infty) \text{ for any value of } y\right]$$

For $y = -6 \in R$, $x = -\frac{1}{3} \notin [0, \infty)$.

Thus, f(x) is not onto.

Hence, f(x) is not invertible.

Since, $x \geq 0$, $\therefore \dfrac{\sqrt{y+6}-1}{3} \geq 0$

$\Rightarrow \quad \sqrt{y+6}-1 \geq 0$

$\Rightarrow \quad \sqrt{y+6} \geq 1$

$\Rightarrow \quad y+6 \geq 1$

$\Rightarrow \quad y \geq -5.$

We redefine,

f: $[0, \infty) \rightarrow [-5, \infty)$,

which makes $f(x) = 9x^2 + 6x - 5$ an onto function.

Now, $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$

$\Rightarrow (3x_1 + 1)^2 = (3x_2 + 1)^2$

$\Rightarrow [(3x_1 + 1) + (3x_2 + 1)][(3x_1 + 1) - (3x_2 + 1)]$

$\Rightarrow [3(x_1 + x_2) + 2][3(x_1 - x_2)] = 0$

$\Rightarrow x_1 = x_2$

$[\because 3(x_1 + x_2) + 2 > 0]$

Thus, f(x) is one-one.

$\therefore$ f(x) is bijective, hence f is invertible

and $f^{-1}: [-5, \infty) \rightarrow [0, \infty)$

$f^{-1}(y) = \dfrac{\sqrt{y+6}-1}{3}$

Assertion and Reason Answers-

1. (a) Both A and R are true and R is the correct explanation of A.

2. (a) Both A and R are true and R is the correct explanation of A.

Case Study Answers-

1. Answer :

(i) (a) R - {2}

Solution:

For f(x) to be defined x - 2; ≠ 0 i.e., x; ≠ 2.

∴ Domain of f = R - {2}

(ii) (b) R - {2}

Solution:

Let y = f(x), then $y = \frac{x-1}{x-2}$

$\Rightarrow$ xy - 2y = x - 1 $\Rightarrow$ xy - x = 2y -

$$\Rightarrow x = \frac{2y-1}{y-1}$$

Since, x ∈∈ R - {2}, therefore y ≠ 1

Hence, range of f = R - {1}

(iii) (d) $\frac{x}{x-2}$

Solution:

We have, g(x) = 2f(x) - 1

$$= 2\left(\frac{x-1}{x-2}\right) - 1 = \frac{2x-2-x+2}{x-2} = \frac{x}{x-2}$$

(iv) (a) One-one

Solution:

We have, $g(x) = \frac{x}{x-2}$

Let $g(x_1) = g(x_2) \Rightarrow \frac{x_1}{x_1-2} = \frac{x_2}{x_2-2}$

$\Rightarrow x_1x_2 - 2x_1 = x_1x_2 - 2x_2 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

Thus, $g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

Hence, g(x) is one-one.

(v) (c) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

2. Answer :

(i) (a) Reflexive

Solution:

Clearly, (1, 1), (2, 2), (3, 3), ∈ R. So, R is reflexive on A.

Since, (1, 2) ∈ R but (2, 1) ∉ R. So, R is not symmetric on A.

Since, (2, 3), ∈ R and (3, 1) ∈ R but (2, 1) ∉ R. So, R is not transitive on A.

(ii) (b) Symmetric

Solution:

Since, (1, 1), (2, 2) and (3, 3) are not in R. So, R is not reflexive on A.

Now, (1, 2) ∈ R ⇒ (2, 1) ∈ R and (1, 3) ∈ R ⇒ (3, 1) ∈ R. So, R is symmetric,

Clearly, (1, 2) ∈ R and (2, 1) ∈ R but (1, 1) ∉ R. So, R is not transitive on A.

(iii) (c) Transitive

Solution:

We have, R = {(x, y): y = x + 5 and x < 4}, where x, y ∈ N.

∴ R = {(1, 6), (2, 7), (3, 8)}

Clearly, (1, 1), (2, 2) etc. are not in R. So, R is not reflexive.

Since, (1, 6) ∈ R but (6, 1) ∉ R. So, R is not symmetric.

Since, (1, 6) ∈ R and there is no order pair in R which has 6 as the first element.

Same is the case for (2, 7) and (3, 8). So, R is transitive.

(iv) (d) Equivalence

Solution:

We have, R = {(x, y): 3x - y = 0}, where x, y ∈ A = {1,2, ..., 14}.

∴ R = {(1, 3), (2, 6), (3, 9), (4, 12)}

Clearly, (1, 1) ∉ R. So, R is not reflexive on A.

Since, (1, 3) ∈ R but (3, 1) ∉ R. So, R is not symmetric on A.

Since, (1, 3) ∈ Rand (3, 9) ∈ R but (1, 9) ∉ R. So, R is not transitive on A.

(v) (d) Equi0076alence

Solution:

Clearly, (1, 1), (2, 2), (3, 3) ∈ R. So, R is reflexive on A.

We find that the ordered pairs obtained by interchanging the components of ordered pairs in R are also in R. So, R is symmetric on A. For 1, 2, 3 ∈ A such that (1, 2) and (2, 3) are in R implies that (1, 3) is also, in R. So, R is transitive on A. Thus, R is an equivalence relation.