MATHEMATICS

Chapter 4: PRINCIPLE OF MATHEMATICAL INDUCTION

PRINCIPLE OF MATHEMATICAL INDUCTION

Top Concepts

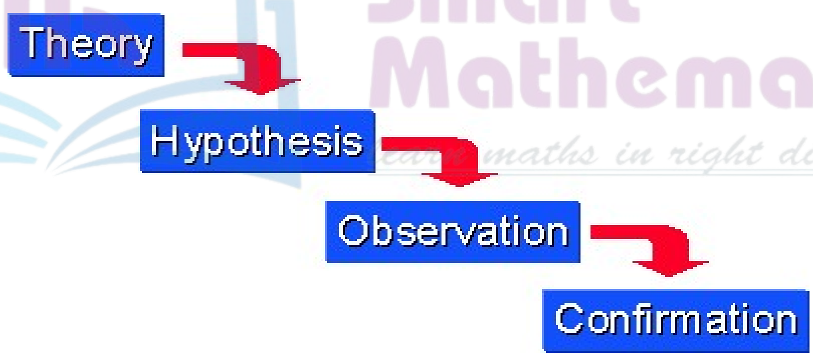
1. There are two types of reasoning—**deductive** and **inductive**.

2. In deduction, given a statement to be proven which is often called a conjecture or a theorem, validdeductive steps are derived and a proof may or may not be established.

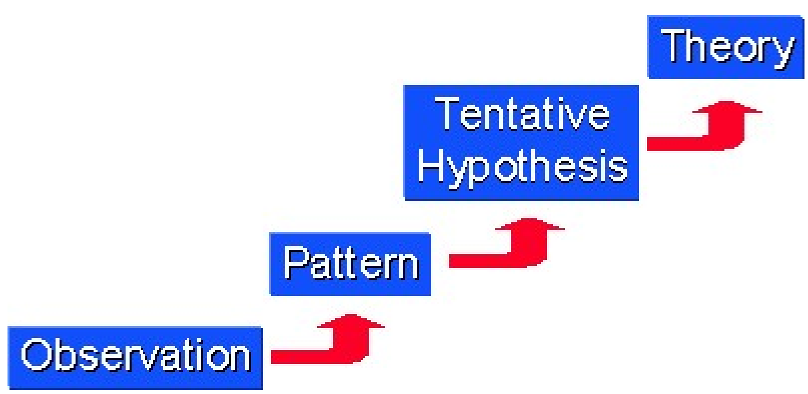
3. Deduction is the application of a general case to a particular case.

4. Inductive reasoning depends on working with each case and developing a conjecture by observingincidence till each and every case is observed.
5. Induction is the generalisation from particular cases or facts.

6. A deductive approach is known as a 'top–down approach'. Given the theorem which is narrowed down to specific *hypotheses* then to *observation*. Finally, the hypotheses is tested with specific data to get the *confirmation* (or not) of original theory.



7. Inductive reasoning works the other way—moving from specific observations to broader generalisations and theories. Informally, this is known as a 'bottom–up approach'.



8. To prove statements or results formulated in terms of n, where n is a positive integer, a principle basedon inductive reasoning called the **Principle of Mathematical Induction (PMI)** is used.

9. PMI is one such tool which can be used to prove a wide variety of mathematical statements. Each ofsuch statements is assumed as
P(n) associated with a positive integer n for which the correctness of the case n = 1 is examined.Then, assuming the truth of P(k) for some positive integer k, the truth of
P(k + 1) is established.

10. Let p(n) denote a mathematical statement such that
(1) p(1) is true.
(2) p(k + 1) is true whenever p(k) is true.
Then, the statement is true for all natural numbers n by PMI.

11. PMI is based on the Peano's Axiom.

12. PMI is based on a series of well-defined steps, so it is necessary to verify all of them.

13. PMI can be used to prove the equality, inequalities and divisibility of natural numbers.

Key Formulae

1. Sum of n natural numbers: 1 + 2 + 3 + ... + n = $\frac{n(n+1)}{2}$
2. Sum of n2 natural numbers: $1^2+2^2+3^2+\ldots\ldots n^2 = \frac{n(n+1)(2n+1)}{6}$
3. Sum of odd natural numbers: 1 + 3 + 5 + 7...... + (2n − 1) = n^2
4. Steps of PMI
   1. Denote the given statement in terms of n by P(n).
   2. Check whether the proposition is true for n = 1.
   3. Assume that the proposition result is true for n = k.
   4. Using p(k), prove that the proposition is true for p(k + 1).

5. Rules of inequalities
   a. If a < b and b < c, then a < c.
   b. If a < b, then a + c < b + c.
   c. If a < b and c > 0 which means c is positive, then ac < bc.
   d. If a < b and c < 0 which means c is positive, then ac > bc.

MIND MAP : LEARNING MADE SIMPLE CHAPTER - 4

Principle of Mathematical Induction

Induction

Specific Instances to Generalisation
e.g.: Rohit eats food. Vikas eats food. Rohit and Vikas are men.Then all men eat food. Statement is true for $n=1, n=k$ and $n=k+1$, then, the statement is true for all natural numbers n.

Deduction

Generalisation of Specific Instance
e.g.: Rohit is a man and all men eat food, therefore, Rohit eats food.

Steps for Principle of Mathematical Induction Proof

Step1: Let P(n) be a result or statement formulated in terms of n in a given equation
Step2: Prove that P(1) is true.
Step3: Assume that P(k) is true.
Step4: Using step 3 , prove that P(k+1) is true.
Step5: Thus, P(1) is true and P(k+1) is true whenever P(k) is true
Hence,by the principle of mathematical induction,P(n) is true for all natural numbers n.

Example

Ex: Prove that $2^n > n$ for all positive integer n.
Solution: Step1: Let P(1):$2^n > n$
Step2: When $n=1$, $2^1 > 1$.Hence P(1) is true
Step3: Assume that P(k) is true for any positive integer k, $2^k > k$
Step4: We shall now prove that P(k+1) is true
Multiplying both sides of step(3) by 2 , we get
$2.2^k > 2k$
$\Rightarrow 2^{k+1} > 2k$
$\Rightarrow 2^{k+1} = k+k$
$\Rightarrow 2^{k+1} > k+1$ (since $k>1$)
Therefore, P(k+1) is true when P(k) is true Hence, by P.M.I., P(n) is true for every positive integer n .

Important Questions

Multiple Choice questions-

Question 1. For all n ∈ N, $3n^5 + 5n^3 + 7n$ is divisible by

(a) 5

(b) 15

(c) 10

(d) 3

Question 2. {1 – (1/2)}{1 – (1/3)}{1 – (1/4)} ……. {1 – 1/(n + 1)} =

(a) 1/(n + 1) for all n ∈ N.

(b) 1/(n + 1) for all n ∈ R

(c) n/(n + 1) for all n ∈ N.

(d) n/(n + 1) for all n ∈ R

Question 3. For all n ∈ N, $3^{2n} + 7$ is divisible by

(a) non of these

(b) 3

(c) 11

(d) 8

Question 4. The sum of the series 1 + 2 + 3 + 4 + 5 + ………..n is

(a) n(n + 1)

(b) (n + 1)/2

(c) n/2

(d) n(n + 1)/2

Question 5. The sum of the series $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots n^2$ is

(a) n(n + 1) (2n + 1)

(b) n(n + 1) (2n + 1)/2

(c) n(n + 1) (2n + 1)/3

(d) n(n + 1) (2n + 1)/6

Question 6. For all positive integers n, the number $n(n^2 - 1)$ is divisible by:

(a) 36

(b) 24

(c) 6

(d) 16

Question 7. If n is an odd positive integer, then $a^n + b^n$ is divisible by :

(a) $a^2 + b^2$

(b) a + b

(c) a – b

(d) none of these

Question 8. n(n + 1) (n + 5) is a multiple of _____ for all n ∈ N

(a) 2

(b) 3

(c) 5

(d) 7

Question 9. For any natural number n, $7^n – 2^n$ is divisible by

(a) 3

(b) 4

(c) 5

(d) 7

Question 10. The sum of the series $1^3 + 2^3 + 3^3 + ...n^3$ is

(a) $\{(n + 1)/2\}^2$

(b) $\{n/2\}^2$

(c) n(n + 1)/2

(d) $\{n(n + 1)/2\}^2$

Very Short:

1.

Short Questions:

1. For every integer n, prove that 7n – 3n divisible by 4.

2. Prove that n (n + 1) (n + 5) is multiple of 3.

3. Prove that $10^{2n-1} + 1$ is divisible by 11.

4. Prove that $\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots.\left(1+\frac{1}{n}\right) = (n+1)$

5. Prove 1.2 + 2.3 + 3.4 + _ _ _ + n $(n+1) = \frac{n(n+1)(n+2)}{3}$

Long Questions:

1. Prove $(2n+7) < (n+3)^2$
2. Prove that:

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+..+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

3. Prove $1.2 + 2.22 + 3.23 + ... + n.2^n = (n-1)^{2n+1} + 2$
4. Prove that $2.7^n + 3.5^n - 5$ is divisible by 24 $\forall\, n \in N$.
5. Prove that $41^n - 14^n$ is a multiple of 27.

Answer Key:

MCQ:

1. (b) 15
2. (a) 1/(n + 1) for all n ∈ N.
3. (d) 8
4. (d) n(n + 1)/2
5. (d) n(n + 1) (2n + 1)/6
6. (c) 6
7. (b) a + b
8. (b) 3
9. (c) 5
10. (d) $\{n(n+1)/2\}^2$

Very Short Answer:

1. $\left(\frac{\pi}{32}\right)^c$
2. 39°22"30""
3. $\frac{5\pi}{12} cm$
4. $\sqrt{3}$
5. $\frac{-1}{\sqrt{2}}$
6. $2-\sqrt{3}$

7. $\frac{-4}{5}$

8. $45°$

9. 2 sin8θ cos4θ

10. sin 6x – sin2x

Short Answer:

1. P(n) : $7^n - 3^n$ is divisible by 4

For n = 1

P(1) : $7^1 - 3^1 = 4$ which is divisible by Thus, P(1) is true

Let P(k) be true

$7^k - 3^k$ *is* divisible by 4

$7^k - 3^k = 4\lambda$, where $\lambda \in N$ (i)

we want to prove that P (k+1) is true whenever P(k) is true

$7^{k+1} - 3^{k+1} = 7^k.7 - 3^k.3$

$= (4\lambda + 3^k).7 - 3^k.3$(from i)

$= 28\lambda + 7.3^k - 3^k.3$

$= 28\lambda + 3^k(7-3)$

$= 4(7\lambda + 3^k)$

Hence

$7^{k+1} - 3^{k+1}$ is divisible by 4

thus P (k+1) is true when P(k) is true.

Therefore by P.M.I. the statement is true for every positive integer n.

2.

$P(n): n(n+1)(n+5)$ is multiple of 3

for n=1

P(1) : 1(1+1) (1+5) = 12 is multiple of 3

let P(k) be true

P(k) : K (k+1) (k+5) is muetiple of 3

$\Rightarrow$ k(k+1)(k+5)=3λ where $\lambda \in N$ (i)

we want to prove that result is true for n=k+1

P(k+1) : (k+1)(k+2)(k+6)

$\Rightarrow$ (K+1)(k+2)(k+6)= $\left[(k+1)(k+2)\right](k+6)$

= k(k+1)(k+2)+6(k+1)(k+2)

=k(k+1)(k+5-3)+6(k+1)(k+2)

=k(k+1)(k+5)-3k(k+1)+6(k+1)(K+2)

=k(k+1)(k+5)+(k+1)$\left[6(k+2)-3k\right]$

=k(k+1)(k+5)+(k+1)(3k+12)

=k(k+1)(k+5)+3(k+1)(k+4)

=3λ+3 (k+1)(k+4) (from i)

=3$\left[\lambda+(K+1)(K+4)\right]$ which is multiple of three

Hence P(k+1) is multiple of 3 .

3.

$P(n): 10^{2n-1}+1$ is divisible by 11

for n=1

P(1) $= 10^{2\times1-1}+1 = 11$ is divisible by 11 Hence result is true for n=1

let P(k) be true

P(k) : $10^{2k-1}+1$ is divisible by 11 1

$\Rightarrow$ $10^{2k-1}+1 = 11\lambda$ where $\lambda \in N$(i)

we want to prove that result is true for n= k+

$=10^{2(k+1)-1}+1 = 10^{2k+2-1}+1$

$=10^{2k+1}+1$

$=10^{2k}.10^{1}+1$

$=(110\lambda-10).10+1$(*from* i)

$=1100\lambda-100+1$

$=1100\lambda-99$

$=11(100\lambda-9)$ *is* divisible by 11

Hence by P.M.I. P (k+1) is true whenever P(k) is true.

4.

$let P(n): \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right) = (n+1)$

for n=1

P(1): $\left(1+\frac{1}{1}\right) = (1+1) = 2$

which is true

let P(k) be true

$$P(k) : \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{k}\right) = (k+1)$$

we want to prove that P(k+1) is true

$$P(k+1): \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)..\left(1+\frac{1}{k+1}\right) = (k+2)$$

$$L.H.S. = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)....\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right)$$

$$=(k+1)\left(1+\frac{1}{k+1}\right) \quad [from(1)]$$

$$= (k+1)\left(\frac{k+1+1}{K+1}\right)$$

$$=(K+2)$$

thus P(k+1) is true whenever

P (K) is true.

5.

$$p(n):1.2+2.3+--n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for $n=1$

$$p(1):1(1+1) = \frac{1(1+1)(1+2)}{3}$$

$$p(1) = 2 = 2$$

hence $p(1)$ be true

$$p(k):1.2+2.3+---+k(k+1) = \frac{k(k+1)(k+2)}{3} ....(i)$$

we want to prove that

$p(k+1)$:

$$1.2+2.3+---+(k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

$L.H.S.$

$$=1.2+2.3+---+k(k+1)+(k+1)(k+2)$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{(k+1)(k+2)}{1} \quad [from(i)]$$

$$\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$$

$$\frac{(k+1)(k+2)[k+3]}{3}$$

hence $p(k+1)$ is true whenenes $p(k)$ is true

Long Answer:

1.

$p(n):(2n+7)<(n+3)^2$

for $n=1$

$9<(4)^2$

$9<16$

which is true

let $p(k)$ be true

$(2k+7)<(k+3)^2$

now

$2(k+1)+7=(2k+7)+2$

$<(k+3)^2+2=k^2+6k+11$

$<k^2+8k+16=(k+4)^2$

$=(k+3+1)^2$

$\therefore p(k+1): 2(k+1)+7<(k+1+3)^2$

$\Rightarrow p(k+1)$ is true, when ever $p(k)$ is true

hence by PMI $p(k)$ is true for all $n\in N$

2.

$$p(n):\frac{1}{1.4}+\frac{1}{4.7}+---+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

for $n=1$

$$p(1):\frac{1}{(3-2)(3+1)}=\frac{1}{(3+1)}=\frac{1}{4}$$

which is true

let $p(k)$ be true

$$p(k):\frac{1}{1.4}+\frac{1}{4.7}+---+\frac{1}{(3k-2)(3k+1)}=\frac{k}{(3k+1)}\text{.........}(i)$$

we want to prove that $p(k+1)$ is true

$$p(k+1): \frac{1}{1.4}+\frac{1}{4.7}+---+\frac{1}{(3k+1)(3k+4)}=\frac{k+1}{(3k+4)}$$

$L.H.S.$

$$=\frac{1}{1.4}+\frac{1}{4.7}+---+\frac{1}{(3k-2)(3k+1)}+\frac{1}{(3k+1)(3k+4)}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)} \qquad [from.......(i)]$$

$$=\frac{k(3k+4)+1}{(3k+1)(3k+4)}$$

$$=\frac{3k^2+4k+1}{(3k+1)(3k+4)}=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$p(k+1)$ is true whenever $p(k)$ is true.

3.

$p(n): 1.2+2.2^2+3.2^3+---+n.2^n=(n-1)2^{n+1}+2$

$p(n): 1.2+2.2^2+3.2^3+---+n.2^n=(n-1)2^{n+1}+2$

for $n=1$

$p(1): 1.2^1=(1-1)2^2+2$

$2=2$ which is true

let $p(k)$ be true

$p(k): 1.2+2.2^2+---+k.2^k=(k-1)2.^{k+1}+2..........(i)$

we want to prove that $p(k+1)$ is true

$p(k+1): 1.2+2.2^2+---+(k+1)2^{k+1}=k.2^{k+2}+2$

$L.H.S.$

$1.2+2.2^2+---+k.2^k+(k+1)2^{k+1} \qquad [\text{from}...........(i)]$

$=(k-1)2^{k+1}+2+(k+1)2^{k+1}$ c

$=2^{k+1}(k-1+k+1)+2$

$=2^{k+2}k+2$

This $p(k+1)$ is true whenever $p(k)$ is true

4. P(n) : $2.7^n + 3.5^n - 5$ is divisible by 24

for n = 1

P (1) : $2.7^1 + 3.5^1 - 5 = 24$ is divisible by 24

Hence result is true for n = 1

Let P (K) be true

P (K) : $2.7^K + 3.5^K - 5$

$\Rightarrow 2.7^K + 3.5^K - 5 = 24\lambda$ when $\lambda \in N$

we want to prove that P (K+!) is True whenever P (K) is true

$2.7^{K+1} + 3.5^{K+1} - 5 = 2.7^K.7^1 + 3.5^K.5^1 - 5$

$= 7\left[2.7^K + 3.5^K - 5 - 3.5^K + 5\right] + 3.5^K.5^1 - 5$

$= 7\left[24\lambda - 3.5^K + 5\right] + 15.5^K - 5$ (from i)

$= 7 \times 24\lambda - 21.5^K + 35 + 15.5^K - 5$

$= 7 \times 24\lambda - 6.5^K + 30$

$= 7 \times 24\lambda - 6(5^K - 5)$

$= 7 \times 24\lambda - 6.4p$ $\left[\because 5^K - 5 \text{ is multiple of } 4\right]$

$= 24(7\lambda - p)$, 24 is divisible by 24

Hence by P M I p (n) is true for all n $\in$ N.

5. P (n) : $41^n - 14^n$ is a multiple of 27

for n = 1

P (1) : $41^1 - 14 = 27$, which is a multiple of 27

Let P (K) be True

P (K) : $41^K - 14^K$

$\Rightarrow 41^K - 14^K = 27\lambda$, where $\lambda \in N$

we want to prove that result is true for n = K + 1

$41^{K+1} - 14^{k+1} = 41^K.41 - 14^K.14$

$= (27\lambda + 14^K).41 - 14^K.14$ (*from* i)

$= 27\lambda.41 + 14^K.41 - 14^K.14$

$= 27\lambda.41 + 14^K(41 - 14)$

$= 27\lambda.41 + 14^K(27)$

$= 27(41\lambda + 14^K)$ is a multiple of 27

Hence by PMI p (n) is true for ace n $\in$ N.