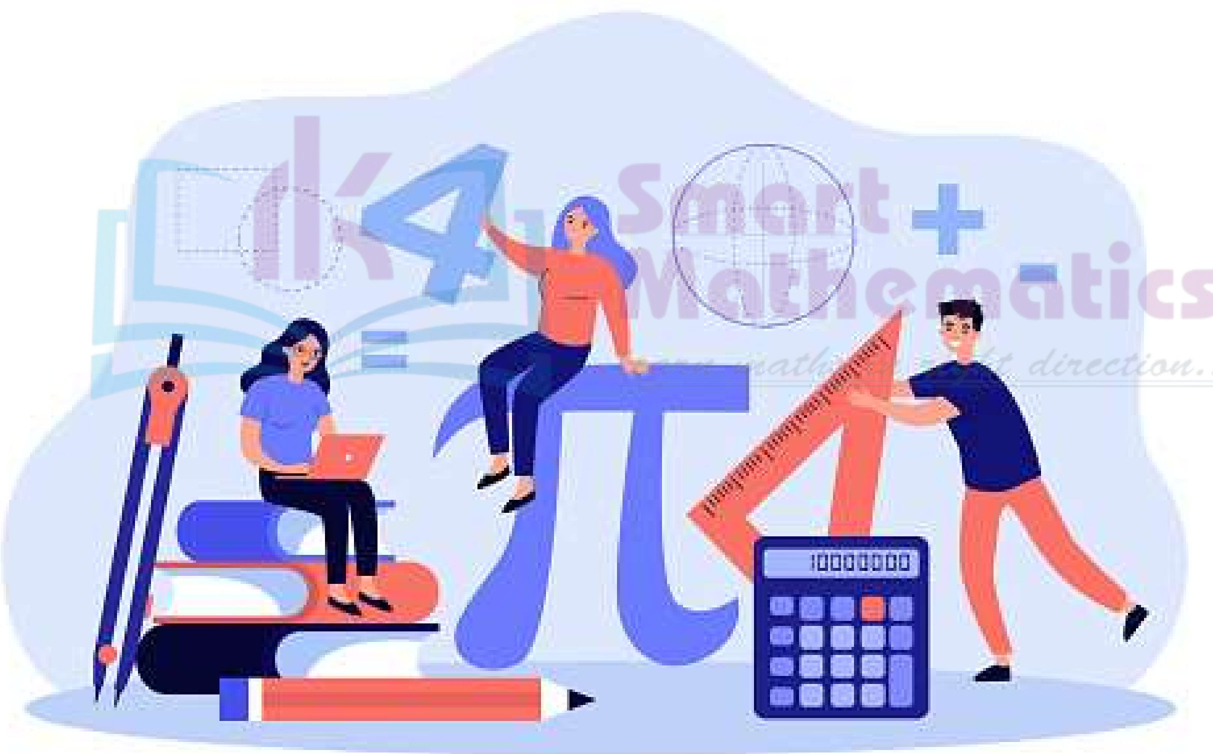
MATHEMATICS

Chapter 8: BINOMIAL THEOREM



BINOMIAL THEOREM

Key Concepts

1. A binomial expression is an algebraic expression having two terms. Examples: (a+b), (a − b) etc.

2. The expansion of a binomial for any positive integral n is given by Binomial Theorem. Thebinomial theorem says that
$(x + y)^n = x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + ---- + {}^nC_r\, x^{n-r}y^r + ----- + {}^nC_{n-1}xy^{n-1} + {}^nc_ny^n$

In summation, notation $(x + y)^n = \sum_{k=0}^{n} {}^nC_k\, x^{n-k}y^k.$

3.
(i) In the binomial expansion of $(x + y)^n$, the number of terms is (n+1), i.e. one more than the exponent.
(ii) The exponent of x goes on decreasing by unity and y increases by unity. Exponent of x is n inthe first term, (n − 1) in the second term etc. ending with zero in the last term.
(iii) The sum of the indices of x and y is always equal to the index of the expression.

4. The coefficient ${}^nC_r$, which is the number of combinations of n objects taken r at a time, occurring inthe binomial theorem are known as binomial coefficients.

5. Binomial coefficients when arranged in the form given below is known as Pascal's Triangle.

Index	Coefficients
0	${}^0C_0$ (=1)
1	${}^1C_0$ (=1) ${}^1C_1$ (=1)
2	${}^2C_0$ (=1) ${}^2C_1$ (=2) ${}^2C_2$ (=1)
3	${}^3C_0$ (=1) ${}^3C_1$ (=3) ${}^3C_2$ (=3) ${}^3C_3$ (=1)
4	${}^4C_0$ (=1) ${}^4C_1$ (=4) ${}^4C_2$ (=6) ${}^4C_3$ (=4) ${}^4C_4$ (=1)
5	${}^5C_0$ (=1) ${}^5C_1$ (=5) ${}^5C_2$ (=10) ${}^5C_3$ (=10) ${}^5C_4$ (=5) ${}^5C_5$ (=1)

6. The array of numbers arranged in the form of triangle with 1 at the top vertex and running down thetwo slanting sides is known as Pascal’s triangle, named after the French mathematician called Blaise Pascal. It is also known as Meru Prastara by Pingla.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

7. Pascal’s triangle is a special triangle of numbers. It has an infinite number of rows. Pascal’striangle is a storehouse of patterns.

8. In order to construct the elements of the following rows, add the number directly above and to the left with the number directly above and to the right to find a new value. If either number to the rightor left is not present, then substitute a zero in its place.

9. Using binomial theorem for non-negative index

$(x - y)^n = [x + (-y)]^n$

$(x - y)^n = {}^nC_0x^n - {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 - {}^nC_3x^{n-3}y^3 + \ldots + (-1)^n\,{}^nC_ny^n.$

In summation notation $(x - y)^n = \sum_{k=0}^{n}(-1)^k\,{}^nC_kx^{n-k}y^k$

10. Binomial theorem can be used to expand the trinomial by applying the binomial expansion twice.

11. The general term in the expansion of $(x + y)^n$ is $T_{k+1} = {}^nC_k\,x^{n-k}\,y^k$.

12. General term in the expansion of $(x-y)^n$ is $T_{k+1} = {}^nC_k\,(-1)^k\,x^{n-k}\,y^k$.

13. If n is odd , then $\{(x+y)^n+(x-y)^n\}$ and $\{(x+y)^n-(x-y)^n\}$ both have the same number of terms equal to $\frac{n+1}{2}$ whereas if n is even, then $\{(x+y)^n+(x-y)^n\}$ has $\left(\frac{n}{2}+1\right)$ terms and $\{(x+y)^n-(x-y)^n\}$ has $\left(\frac{n}{2}\right)$ terms.

14. If n is even, there is only one middle term in the expansion of $(x+y)^n$ and will be the $\left(\frac{n}{2}+1\right)^{th}$ term.

15. If n is odd, then there are two middle terms in the expansion of $(x+y)^n$ and they are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ term.

16. In the expansion $\left(x+\frac{1}{x}\right)^{th}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{th}$, i.e. $(n+1)^{th}$ term as 2n is even.

17. In the binomial expansion of $(x+y)^n$, the k^{th} term from the end is $((n+1)-k+1)=(n-k+2)^{th}$ term from the beginning.

18. Coefficient of $(k+1)^{th}$ term in the binomial expansion of $(1+x)^n$ is ${}^nC_k$.

19. Coefficient of x^k term in the binomial expansion of $(1+x)^n$ is ${}^nC_k$.

20. Coefficient of x^k term in the binomial expansion of $(1-x)^n$ is $(-1)^k\,{}^nC_k$.

21. Coefficient of $(k+1)^{th}$ term in the binomial expansion of $(1-x)^n$ is $(-1)^k\,{}^nC_k$.

Some Important conclusions

1. $(1+x)^n = \sum_{i=0}^{n} {}^nC_i x^i$

   This is the expansion of $(1+x)^n$ in ascending powers of x.

2. $(1+x)^n = \sum_{i=0}^{r} {}^nC_i x^{n-i}$

   This is the expansion of $(1+x)^n$ in descending powers of x.

3. $(1-x)^n = \sum_{i=0}^{n} (-1)^i \; {}^nC_i x^i$

4. $(x+y)^n + (x-y)^n = 2\left[{}^nC_0 x^n y^0 + {}^nC_2 x^{n-2} y^2 + {}^nC_4 x^{n-4} y^4 + \ldots\right]$

   and

   $(x+y)^n - (x-y)^n = 2\left[{}^nC_1 x^{n-1} y^1 + {}^nC_3 x^{n-3} y^3 + {}^nC_5 x^{n-5} y^5 + \ldots\right]$

5. Let S_{odd} and S_{even} denote the sums of odd terms and even terms in the expansion of $(x+y)^n$, then

   1. $(x+y)^n = S_{odd} + S_{even}$ and $(x-y)^n = S_{odd} - S_{even}$
   2. $(x^2-y^2)^n = (S_{odd})^2 - (S_{even})^2$
   3. $4S_{odd}S_{even} = (x-y)^{2n} - (n-y)^{2n}$
   4. $(x+y)^{2n} + (x-y)^{2n} = 2\left[(S_{odd})^2 + (S_{even})^2\right]$

MIND MAP : LEARNING MADE SIMPLE CHAPTER - 8

Binomial Theorem

Binomial Theorem for any Index

If n is negative integer, then n! is not defined. We state bionomial theorem in another form

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + --- + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}b^r + \cdots\cdots + b^n$$

Here, $T_{r+1} = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{r!}a^{n-r}b^r$

General & Middle Terms in Expansion of $(a+b)^n$

The general term of an expansion $(a+b)^n$ is

$T_{r+1} = {}^nC_r\, a^{n-r} b^r,\ 0 \le r \le n, r \in N$

Middle Terms:

1. In $(a+b)^n$, if n is even, then the no. of terms in the expansion is odd. Therefore, there is only one middle term and it is $\left(\frac{n+2}{2}\right)^{th}$ term.
2. In $(a+b)^n$, if n is odd then the no. of terms in the expansion is even. Therefore, there are two middle terms and those are $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms.

Some Special Cases $(a+b)^n$

In the expansion of $(a+b)^n$,

(i) Taking a=x and b=−y, we obtain

$(x-y)^n = {}^nC_0x^n - {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + --- + (-1)^n\, {}^nC_n y^n$

(ii) Taking a=1, b=x, we obtain

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ------- + {}^nC_nx^n$

(iii) Taking a=1, b=−x, we obtain

$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + ---- + (-1)^n\, {}^nC_nx^n$

Definition

If a, b∈R and n∈N, then

$$(a+b)^n = {}^nC_0a^nb^0 + {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + ------ + {}^nC_na^0b^n$$

- **Remarks:** If the index of the binomial is n then the expansion contains n+1 terms.
- In each term, the sum of indices of a and b is always n.
- Coefficients of the terms in binomial expansion equidistant from both the ends are equal.

$$(a-b)^n = {}^nC_0a^nb^0 - {}^nC_1a^{n-1}b^1 + {}^nC_2a^{n-2}b^2 + ------ + (-1)^n\, {}^nC_na^0b^n$$

Binomial Coefficients

The coefficient ${}^nC_0, {}^nC_1, {}^nC_2, ------ {}^nC_n$, in the expansion of $(a+b)^n$ are called binomial coefficients and denoted by $C_0, C_1, C_2 ---- C_x$, respectively

Properties of binomial coefficients:

(i) $C_0 + C_1 + C_2 + ------ + C_n = 2^n$

(ii) $C_0 - C_1 + C_2 - ------ + (-1)^n C_n = 0$

(iii) $C_0 + C_2 + C_4 + ---- = C_1 + C_3 + C_5 + ---- = 2^{n-1}$

(iv) ${}^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$

(v) ${}^nC_{r_1} + {}^nC_{r-1} = {}^{1+n}C_r$

(vi) ${}^nC_r = \frac{n}{r}\, {}^{n-1}C_{r-1}$

Important Questions

Multiple Choice questions-

Question 1. The number $(101)^{100} – 1$ is divisible by

(a) 100

(b) 1000

(c) 10000

(d) All the above

Question 2. The value of -1° is

(a) 1

(b) -1

(c) 0

(d) None of these

Question 3. If the fourth term in the expansion $(ax + 1/x)^n$ is 5/2, then the value of x is

(a) 4

(b) 6

(c) 8

(d) 5

Question 4. The number 111111 …………. 1 (91 times) is

(a) not an odd number

(b) none of these

(c) not a prime

(d) an even number

Question 5. In the expansion of $(a + b)^n$, if n is even then the middle term is

(a) $(n/2 + 1)^{th}$ term

(b) $(n/2)^{th}$ term

(c) n^{th} term

(d) $(n/2 – 1)^{th}$ term

Question 6. The number of terms in the expansion $(2x + 3y – 4z)^n$ is

(a) n + 1

(b) n + 3

(c) {(n + 1) × (n + 2)}/2

(d) None of these

Question 7. If A and B are the coefficient of x^n in the expansion $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then A/B equals

(a) 1

(b) 2

(c) 1/2

(d) 1/n

Question 8. The coefficient of y in the expansion of $(y^2 + c/y)^5$ is

(a) 29c

(b) 10c

(c) $10c^3$

(d) $20c^2$

Question 9. The coefficient of x^{-4} in $(3/2 – 3/x^2)^{10}$ is

(a) 405/226

(b) 504/289

(c) 450/263

(d) None of these

Question 10. If n is a positive integer, then $9^{n+1} – 8n – 9$ is divisible by

(a) 8

(b) 16

(c) 32

(d) 64

Very Short:

1. What is The middle term in the expansion of $(1 + x)^{2n+1}$
2. When is n a positive integer, the no. of terms in the expansion $(x + a)^n$ of is
3. Write the general term $(x^2 - y)^6$
4. In the expansion of $\left(x+\frac{1}{x}\right)^6$, find the 3rd term from the end.
5. Expand $(1 + x)^n$
6. The middle term in the expansion of $(1 + x)^{2n}$ is.

7. Find the no. of terms in the expansions of $(1 - 2x + x^2)^7$
8. Find the coeff of x^5 in $(x + 3)^9$
9. Find the term independent of $x\left(x+\frac{1}{x}\right)^{10}$
10. Expand $(a + b)^n$

Short Questions:

1. Which is larger $(1.01)^{10,000000}$ or 10,000.
2. Prove that:

$$\sum_{r=0}^{n} 3^r \ {}^nC_r$$

3. Using binomial theorem, prove that $6^n - 5n$ always leaves remainder 1 when divided by 25.
4. Find the 13th term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$
5. Find the term independent of x in the expansion of $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}, x > 0$

Long Questions:

1. Find , if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^{n}$ is $\sqrt{6}: 1$
2. The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio 1 : 7 : 42. Find n.
3. The second, third and fourth terms in the binomial expansion $(x + a)^n$ are 240, 720 and 1080 respectively. Find x , a and n.
4. If a and b are distinct integers, prove that a-b is a factor $a^n - b^n$, of whenever n is positive.
5. The sum of the coeff. Of the first three terms in the expansion of $\left(x-\frac{3}{x^2}\right)^{m}$ m being natural no. is 559. Find the term of expansion containing x^3.

Answer Key:

MCQ:

1. (d) All the above
2. (b) -1
3. (b) 6
4. (c) not a prime
5. (a) $(n/2 + 1)^{th}$ term
6. (c) {(n + 1) × (n + 2)}/2
7. (b) 2
8. (c) $10c^3$
9. (d) None of these
10. (d) 64

Very Short Answer:

1. Since (2n + 1) is odd there is two middle term

$i.e\ {}^{2n+1}C_{n}\, x^{n+1}$ and ${}^{2n+1}C_{n+1}\, x^{n}$

2. The no. of terms in the expansion of (x + a) is one more than the index n. i.e. (n + 1).

3.

$T^{r+1} = {}^{6}C_{r}\left(x^2\right)^{6-r}.(-y)^r$

$= {}^{6}C_{r}(x)^{12-2r}.(-1)^r.(y)^r$

4. 3^{rd} term form end = $(6 – 3 + 2)^{th}$ term from beginning

$T_5 = {}^{6}C_{4}(x)^{6-4}.\left(\frac{1}{x}\right)^4$

$= {}^{6}C_{4}x^2.x^{-4}$

$= 15^{x-2}$

$= \frac{15}{x^2}$

5.

$(1+x)^n = 1 + {}^{n}C_{1}(x)^1 + {}^{n}C_{2}(x)^2 + {}^{n}C_{3}(x)^3 + ...x^n$

6.

$^{2n}C_n . x^n$

7.

$\left(1-2x+x^2\right)^7$

$=\left(x^2-2x+1\right)^7$

$=\left[(x-1)^2\right]^7$

$=(x-1)^{14}$

No. of term is 15

8.

$T_{r+1} = {}^9C_r (x)^{9-r} .(3)^r$

Put $9-r=5$

$r=4$

$T_5 = {}^9C_4 (x)^5 .(3)^4$

Coeff of x^5 is ${}^9C_4 (3)^4$

9.

$T_{r+1} = {}^{10}C_r (x)^{10-r} .\left(\frac{1}{x}\right)^r$

$= {}^{10}C_r (x)^{10-r} .(x)^{-r}$

$= {}^{10}C_r (x)^{10-2r}$

Put $10-2r=0$

$r=5$

Independent term is ${}^{10}C_5$

10.

$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + ... + {}^nC_n b^n$

Short Answer:

1.

$(1.01)^{10,00000} = (1+0.01)^{10,00000}$

$= {}^{10,00000}C_0 + {}^{10,00000}C_1(0.01) + \text{other positive term}$

$= 1 + 10,00000 \times 0.01 + \text{other positive term}$

$= 1 + 10,000$

$= 10,001$

Hence $(1.01)^{10,00000} > 10,000$

2.

$\sum_{r=0}^{n} 3^r \; {}^nC_r = \sum_{r=0}^{n} {}^nC_r . 3^r$

$= {}^nC_0 + {}^nC_1 .3 + {}^nC_2 .3^2 + ... + {}^nC_n 3^n$

$\left[\because (1+a)^n = 1 + {}^nC_1 .a + {}^nC_2 a^2 + {}^nC_3 a^3 + ... + a^n\right]$

$= (1+3)^n$

$= (4)^n$

$H.P$

3.

Let $6^n = (1+5)^n$

$= 1 + {}^nC_1 5^1 + {}^nC_2 5^2 + {}^nC_3 5^3 + ... + 5^n$

$= 1 + 5n + 5^2\left({}^nC_2 + {}^nC_3 .5 + ... + 5^{n-2}\right)$

$6^n - 5n = 1 + 25\left({}^nc_2 + {}^nc_3 .5 + ... + 5^{n-2}\right)$

$= 1 + 25k\left[\text{where } k = {}^nc_2 + {}^nc_3 .5 + ...5^{n-2}\right]$

$= 25k + 1$

$H.P$

4.

The general term in the expansion of

$\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ is

$T_{r+1} = {}^{18}C_r (9x)^{18-r}\left(-\frac{1}{3\sqrt{x}}\right)^r$

For 13th term, $r+1=13$

$r=12$

$= {}^{18}C_{12}(9x)^6\left(-\frac{1}{3\sqrt{x}}\right)^{12}$

$= {}^{18}C_{12}(3)^{12}.x^6\left(-\frac{1}{3}\right)^{12}.(x)^{-6}$

$= {}^{18}C_{12}(3)^{12}.(-1)^{12}.(3)^{-12}$

$= {}^{18}C_{12}$

$=18564$

5.

$T_{r+1} = {}^{18}C_r\left(\sqrt[3]{x}\right)^{18-r}\left(\frac{1}{2\sqrt[3]{x}}\right)^r$

$= {}^{18}C_r(x)^{\frac{18-r}{3}}.\left(\frac{1}{2}\right)^r.x^{\frac{-r}{3}}$

$= {}^{18}C_r(x)^{\frac{18-r-r}{3}}.\left(\frac{1}{2}\right)^r$

For independent term $\frac{18-2r}{3}=0$

$r=9$

The req. term is ${}^{18}C_9\left(\frac{1}{2}\right)^9$

Long Answer:

1.

Fifth term from the beginning in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is

$T_{4+1} = {}^nC_4\left(\sqrt[4]{2}\right)^{n-4}.\left(\frac{1}{\sqrt[4]{3}}\right)^4$

$T_5 = {}^nC_4(2)^{\frac{n-4}{4}}.(3)^{-1}......(i)$

How fifth term from the end would be equal to $(n-5+2)$ in term from the beginning

$$T_{(n-4)+1} = {}^{n}C_{n-4}\left(\sqrt[4]{2}\right)^{n-(n-4)} \cdot \left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$$

$$= {}^{n}C_{n-4}(2)^{1}(3)^{\frac{n-4}{4}} \;.......(ii)$$

ATQ $$\frac{{}^{n}C_{4}.(2)^{\frac{n-4}{4}}(3)^{-1}}{{}^{n}C_{n-4}(2)^{1}(3)^{\frac{n-4}{-4}}} = \frac{\sqrt{6}}{1}$$

$$\frac{(2)^{\frac{n-8}{4}}}{(3)^{\frac{-(n-8)}{4}}} = (6)^{\frac{1}{2}}$$

$$(6)^{\frac{n-8}{4}} = (6)^{\frac{1}{2}}$$

$$\frac{n-8}{4} = \frac{1}{2}$$

$$\Rightarrow 2n-16 = 4$$

$$n = 10$$

2.

Let three consecutive terms in the expansion of $(1+a)^{n}$ are $(r-1)^{th}$, r^{th} and $(r+1)^{th}$ term

$$T_{r+1} = {}^{n}C_{r}(1)^{n-r}.(a)^{r}$$

$$T_{r+1} = {}^{n}C_{r}\;(a)^{r}$$

Coefficients are

${}^{n}C_{r-2}$, ${}^{n}C_{r-1}$ and ${}^{n}C_{r}$ respectively

ATQ $\frac{{}^{n}C_{r-2}}{{}^{n}C_{r-1}} = \frac{1}{7}$

$$\Rightarrow n-8r+9 = 0 ....(i)$$

$$\frac{{}^{n}C_{r-1}}{{}^{n}C_{r}} = \frac{7}{42}$$

$$\Rightarrow n-7r+1 = 0 ......(ii)$$

On solving eq. (i) and (ii) we get $n = 55$

3.

$T_2 = 240$

${}^{n}C_{1} x^{n-1}.a = 240 ....(i)$

${}^{n}C_{2} x^{n-2}.a^2 = 720 ....(ii)$

${}^{n}C_{3} x^{n-3}.a^3 = 1080 ....(iii)$

Divide (ii) by (i) and (iii) by (ii)

We get

$\frac{a}{x} = \frac{6}{n-1}$ and $\frac{a}{x} = \frac{9}{2(n-2)}$

$\Rightarrow n = 5$

On solving we get

$x = 2$

$a = 3$

4.

Let $a^n = (a-b+b)^n$

$a^n = (b+a-b)^n$

$= {}^{n}C_{0} b^n + {}^{n}C_{1} b^{n-1}(a-b) + {}^{n}C_{2} b^{n-2}.(a-b)^2 + {}^{n}C_{3} b^{n-3}.(a-b)^3 + ... + {}^{n}C_{n}(a-b)^n$

$a^n = b^n + (a-b)\left[{}^{n}C_{0} b^n + {}^{n}C_{1} b^{n-1}(a-b) + {}^{n}C_{2} b^{n-2}.(a-b)^2 + {}^{n}C_{3} b^{n-3}.(a-b)^3 + ... + {}^{n}C_{n}(a-b)^n\right]$

$a^n - b^n = (a-b)k$

Where

${}^{n}C_{1} b^{n-1} + {}^{n}C_{2} b^{n-2}(a-b) + ... + (a-b)^{n-1} = k$

$H.P$

5. The coeff. Of the first three terms of $\left(x - \frac{3}{x^2}\right)^m$ are ${}^{m}C_{0}, (-3)\,{}^{m}C_{1}$ and $9\ {}^{m}C_{2}$.

Therefore, by the given condition

${}^{m}C_{0} - 3\,{}^{m}C_{1} + 9\,{}^{m}C_{2} = 559$

$1 - 3m + \frac{9m(m-1)}{2} = 559$

On solving we get $m = 12$

$T_{r+1} = {}^{12}C_r (x)^{12-r} \left(\frac{-3}{x^2}\right)^r$

$= {}^{12}C_r (x)^{12-r} . (-3)^r . (x)^{-2r}$

$= {}^{12}C_r (x)^{12-3r} . (-3)^r$

$12 - 3r = 3 \Rightarrow r = 3$, *req*. term is -5940 x^3